Math 1552

Section 10.6: Alternating Series Review

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Review Question:
The series: $$\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{k^2+1}}$$

A. Converges absolutely

B. Converges conditionally

C. Diverges

Countdown

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Section 10.7: Power Series

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Learning Goals

- Recognize the general forms of a power series
- Understand that a power series is an infinite polynomial
- Determine the radius and interval of convergence for a power series
- Differentiate and integrate a power series to obtain a new power series

Power Series

A *power series* is an *infinite polynomial* and a *function* of *x*:

$$\text{Power series in } x: \; f(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$\text{Power series in } x\text{-}c: \; f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$$

Convergence of Power Series

$$\sum_{k=0}^{\infty} a_k (x-c)^k \text{ converges at } x_0$$

$$\text{if } \sum_{k=0}^{\infty} a_k (x_0-c)^k \text{ converges.}$$

The series converges on $\left(x_0, x_1\right)$

if it converges at every point in the interval.

Interval of convergence

The *interval of convergence* of a power series is the set of all values of x for which the series converges.

This interval may be closed, open, or half-open.

Question: On which interval do you think this series converges? *(Why?)*

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

Radius and I.C.

To find the radius of convergence of a power series in **standard form**, use the ratio or root test to find:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{ or } L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

(i) If $L=0$, then $R=\infty$ and I.C. is all real numbers.

(ii) If $L=\infty$, then $R=0$ and I.C. is just $x=c$.

(iii) If L is positive and finite, then $R=1/L$, and the series converges for $|x-c|<R$. You must also check the endpoints.

Cautionary note!

If the power series is not in standard form, you may find the radius using one of these two methods:

1. Rewrite the series in standard form, and then evaluate the limit.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \text{ or } L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

2. Put the *entire series* (with the "x" terms as well!) into the ratio or root test, then solve to find where the resulting limit is less than 1.

Example 1.1:

Find the radius and interval of convergence for the power series.

$$\sum_{k=1}^{\infty} \frac{(x+1)^k}{6^k}$$

Example 1.2:

Find the radius and interval of convergence for the power series.

$$\sum_{k=1}^{\infty} \frac{(4-3x)^k}{\sqrt{2k+5}}$$

Example:

Find the radius and interval of convergence for

$$\sum_{k=0}^{\infty} \frac{2^k}{k^3}(x-1)^k$$

A. R=1/2, I.C.=[1/2,3/2]
B. R=2, I.C.=(-1,3)
C. R=1/2, I.C.=[1/2,3/2)
D. R=2, I.C.=[-1,3]

Differentiation and Integration

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$\int\left(\sum_{k=0}^{\infty} a_k x^k\right)dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$$

The radius and interval of convergence are preserved under termwise differentiation and integration.

Example 2.1:

Find a power series expansion for the function

$$f(x) = \ln(x + 1)$$

For what values of x is this formula valid?

Example 2.2:

Find a power series expansion for the function

$$g(x) = \frac{1}{(1-x)^2}$$

For what values of x is this formula valid? *(Explain briefly.)*

Bonus Problem A:

Evaluate the sum $$\sum_{n=0}^{\infty} \frac{(n+1)}{2^n}$$

Bonus Problem B:

Evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{3^n}$

Hints:

- ❑ See that $n^2 = n(n-1) + n$
- ❑ We have that

$$\frac{d^2}{dx^2}\left[\frac{1}{1-x}\right] = \sum_{n=0}^{\infty} n(n-1)x^{n-2}, x > 0$$

Bonus Problem C:

If $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, show that $\int_0^1 \frac{\tan^{-1}(x)}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$

<u>Challenge example</u>: (A neat trick with power series)

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

Show that $A_{\text{even}}(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} \qquad = \frac{1}{2}\left(A(x) + A(-x)\right)$

$$A_{\text{odd}}(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = \frac{1}{2}\left(A(x) - A(-x)\right)$$

Where do each of the latter two functions converge?

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Sections 10.8 and 10.9: Taylor Polynomials and Taylor Series

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Learning Goals

- Understand the process to finding a Taylor polynomial for a given function and center
- Estimate a function value using Taylor Polynomials and a specified error range
- Recognize standard formulas for basic MacLaurin series
- Manipulate the standard series to find MacLaurin series for other functions
- Appropriately use error terms for alternating and non-alternating Taylor series

Taylor Polynomial

A *Taylor Polynomial* for a continuous function *f* about *x=a* is defined as:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Note that if *a=0*, the formula reduces to:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Example 1:

Find the third-degree Taylor polynomial of the function

$$f(x) = \sqrt{x}$$

in powers of (x-1).

Question: Find a fourth-degree Taylor polynomial for f(x)=cos(x) about x=0.

A. $1-\frac{x^2}{2!}+\frac{x^4}{4!}$

B. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$

C. $x-\frac{x^3}{3!}$

D. $1+x+x^2+x^3+x^4$

Taylor Remainder Term

The remainder term for P_n, where c is some number between a and x, is given by:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

We can find an upper bound for the remainder using the formula:

$$|R_n(x)| \leq \max|f^{(n+1)}(c)|\frac{|x-a|^{n+1}}{(n+1)!}$$

(Over what range of c is the maximum taken?)

Example 2:

Find the maximum error when

$\sqrt{1.5}$ is approximated using a 3rd degree

Taylor polynomial to the function

$f(x)=\sqrt{2-x}.$

Example 3:

Approximate

$$e^{0.2}$$

within an error of at most 0.01.

Taylor Series

A *Taylor series* is an infinite Taylor polynomial:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

In other words, *a Taylor polynomial is the n^{th} partial sum of a Taylor series*.

If *a=0*, a Taylor series is called a *MacLaurin series*.

Common MacLaurin Series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, x \in \Re$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, x \in \Re$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, x \in \Re$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, |x| < 1$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, |x| < 1$$

Example 4.1:

Find a MacLaurin series for the following function:

$$f(x) = \frac{\sin(5x)}{x}$$

(Where does it converge?)

Example 4.2:

Find a MacLaurin series for the following function:

$$g(x) = \frac{4x}{2+x}$$

(Where does it converge?)

Example: Find a MacLaurin series for $f(x) = \cos(2x)$

1. $2\sum_k (-1)^k \dfrac{x^{2k}}{(2k)!}$
2. $\sum_k (-1)^k \dfrac{x^{2k+2}}{k!}$
3. $\sum_k (-1)^k \dfrac{2^k x^{2k}}{(2k)!}$
4. $\sum_k (-1)^k \dfrac{4^k x^{2k}}{(2k)!}$

Recall: Differentiation and Integration of Power Series

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$\int\left(\sum_{k=0}^{\infty} a_k x^k\right)dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$$

The radius and interval of convergence are preserved under differentiation and Integration.

Example 5:

Find a MacLaurin Series for the function $f(x) = x^2 \tan^{-1}(x)$

__Example__: Find a power series for $\int_0^x \cos\left(t^2\right) dt$

1. $\sum_k (-1)^k \dfrac{x^{4k+1}}{(4k+1)(2k)!}$
2. $\sum_k (-1)^k \dfrac{x^{4k+4}}{(4k+4)(2k)!}$
3. $\sum_k (-1)^k \dfrac{x^{4k^2+1}}{(4k^2+1)(2k)!}$
4. $\sum_k (-1)^k \dfrac{x^{2k+1}}{(2k+1)(2k)!}$

Example 6:

Estimate $\int_0^{1/2} \cos(x^3)\,dx$ within an error range of 0.001.